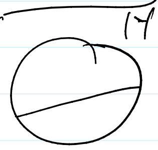
We consider the upper bound of $ex(n, C_{2k})$ $\forall k \geq 2$.

<u>Thm</u> (Bondy-Simonovits) $\exists$ a constant $c>0$ s.t. $\forall k \geq 2$, $ex(n, C_{2k}) \leq ckn^{1+1/k}$.

<u>Rmk</u> The original proof gives $c = 100$.

<u>Lemma</u> (A-B path) Let H be a graph consisting of a cycle with a chord, and let (A, B) be a non-trivial partition of $V(H)$. Then for $\forall$ $\ell < |V(H)|$, there is an <u>(A,B)-path</u> of length ℓ in H, unless ℓ is even and H is bipartite with the bipartition (A, B).

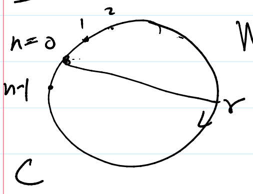
(1)



θ-graph

$\Downarrow$

<u>Pf</u>. Let the cycle $C = 012\cdots(n-1)0$ with chord $0r$.



We take indices under modulus n.

Let $\chi: V(H) \to \{0,1\}$ by $\chi(i) = \begin{cases} 1, & i \in A \\ 0, & i \in B. \end{cases}$

Let $P = \{p \in \mathbb{Z}_n^+ : \forall i,\ \chi(i) = \chi(i+p)\}$

So if $\ell \notin P$, then we can find an (A,B)-path of length ℓ using only the edges of C.

It suffices for us to consider $\ell \in P$.

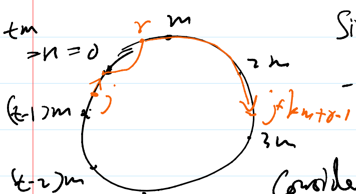
Let $m \in P$ be the smallest positive integer in P.

Then $m \mid n$. (EX).

For all ℓ with $\ell \nmid m$, there exists some (A,B)-path of length ℓ (by definition of m)

Thus, we only need to consider $\boxed{\ell = km}$.

<u>Case 1</u>. The chord $0r$ satisfies that $1 < r \leq m$.



Since $m \nmid (m+r-1)$, there is some $-m < j \leq 0$ s.t. $\chi(j) \neq \chi(j+m+r-1)$
$= \chi(j+km+r-1)$

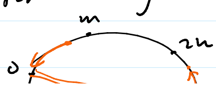
Consider the path

$j(j+1)\cdots 0r(r+1)\cdots(j+m+r-1)\cdots(j+km+r-1)$

This is an (A,B)-path of length $km = \ell$. $\boxtimes$

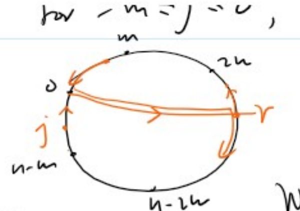
<u>Case 2</u> $m < r < n-m$

For $-m \leq j \leq 0$, define 2 paths



$P_j = j(j+1)\cdots 0r(r-1)\cdots(r-j+m+1)$

for $-m \le j \le 0$, define …



$P_j = j(j+1)\cdots 0\, r(r-1)\cdots(r-j+m+1)$

$Q_j = (m+j)(m+j-1)\cdots 0\, r(r+1)\cdots(r-j-1)$

We see both paths have length m.

(2.1) Suppose $\exists$ some j with $-m \le j \le 0$ such that P_j or Q_j is an (A,B)-path.

Then we can extend it to an (A,B)-path of length $km = \ell$ by adding a subpath of length m at a time.

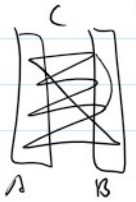
(2.2) We may assume $\forall\, -m \le j \le 0$, P_j and Q_j are not (A,B)-paths

$\Rightarrow$ $\chi(j) = \chi(r-j+m+1)$ & $\chi(m+j) = \chi(r-j-1)$ $\forall\, -m \le j \le 0$.

$\Rightarrow$ $\chi(r-j+1) = \chi(r-j-1)$ $\forall\, -m \le j \le 0$

$\Rightarrow$ $\chi(i) = \chi(i+2)$ $\forall\, i$

$\Rightarrow$ $m = 2$ and the vertices of C alternate between A and B ($2 \mid n$)



If the chord xr is in the same part, then one can check that H contains (A,B)-paths of all possible length.

O.W. the chord xr is between (A,B). Then (A,B) is the bipartition of H. $\boxtimes$

Pf 1 (Bondy–Simonovits) We show $\boxed{\text{Using A-B path Lemma}}$

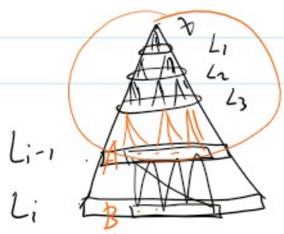
$$ex(n, C_{2k}) \le 2k n^{1+1/k} + 6(k-1)n.$$

Let G be an n-vtx C_{2k}-free graph with more than $2kn^{1+1/k} + 6(k-1)n$ edges.

Then G has a bipartite subgraph H' with $e(H') > kn^{1+1/k} + 3(k-1)n$.

Further, H' has a bipartite subgraph H with $\boxed{\delta(H) > kn^{1/k} + 3(k-1)}$

Let T be a breadth-first search tree (BFS tree) with root x. Let $L_i = \{u \in V(H) : d_H(x,u) = i\}$ for $i \ge 1$.



Since H is bipartite, each L_i is stable.

$\boxed{\text{Claim 1. } e_H(L_{i-1}, L_i) \le (k-1)(|L_{i-1}| + |L_i|) \text{ for each } 1 \le i \le k.}$

Pf. Suppose ... $e(L_{i-1}, L_i) > (k-1)(|L_{i-1}| + |L_i|)$

for each $1 \le i \le k$.

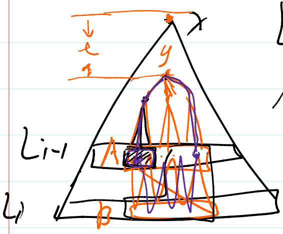
$\boxed{\text{pf}}$ Suppose not, $e(L_{i-1}, L_i) > (k-1)(|L_{i-1}| + |L_i|)$ for some $i \geq 2$.

Then $H(L_{i-1}, L_i)$ has a subgraph H_1 with $\delta(H_1) \geq k$.

Then H_1 has an even cycle C of length at least $2k$ with a chord.

Let $A = V(C) \cap L_{i-1}$ & $B = V(C) \cap L_i$

Let T' be a subtree of the BFS tree T s.t. $A \subseteq V(T')$ and subject to this, T' is minimal.



Let y be the root of T'.

As T' is minimal, y has at least 2 branches.

Let A' be the subset of A containing all vertices from one branch of T'.

$\Longrightarrow$ $A \setminus A' \neq \emptyset$. Let $B' = (A \cup B) \setminus A'$.

$\Longrightarrow$ $\boxed{(A', B') \text{ is NOT a bipartition of } H_1.}$

Let ℓ be the distance between x and y.

$\Longrightarrow$ $\ell < i$ and $2k - 2i + 2\cancel{\ell} \underset{+2}{} < 2k \leq |V(C)|$.

By A-B path Lemma, we can find an (A', B')-path P of length $\underline{2k-2i+2\ell}$ $(+2)$ in H_1 between $a \in A'$ and $b \in B'$.

As $|P|$ is even, $b \in A \setminus A'$.

Let P_a and P_b be the unique paths in T' that connect y to a and b, respectively.

$\Longrightarrow$ a cycle $P_a \cup P_b \cup P$ of length $|P_a| + |P_b| + |P| = 2k$, a contradiction. This proves claim 1. $\boxtimes$

<u>claim 2</u>. $|L_i| \geq n^{1/k} |L_{i-1}|$ for $\forall i \in [k]$.

<u>pf</u>. By induction on i. <u>Base Case $i=1$</u> ✓

For $i \geq 2$, $(kn^{1/k} + 3(k-1))|L_{i-1}| \leq \sum_{v \in L_{i-1}} d_H(v) = e(L_{i-2}, L_{i-1}) + e(L_{i-1}, L_i)$

$$\overset{\text{claim 1}}{\leq} (k-1)\left(|L_{i-2}| + 2|L_{i-1}| + |L_i|\right)$$

$$\leq (k-1)\left(3|L_{i-1}| + |L_i|\right)$$

$\Longrightarrow$ $|L_i| \geq \underline{kn^{1/k}}\, |L_i| > n^{1/k} \cdot |L_i|$. $\boxtimes$

$\geq (k-1)(\ldots|L_{i-1}|\ldots)$

$\Rightarrow |L_i| \geq \frac{kn^{1/k}}{(k-1)}|L_i| > n^{1/k} \cdot |L_i|$. ☒

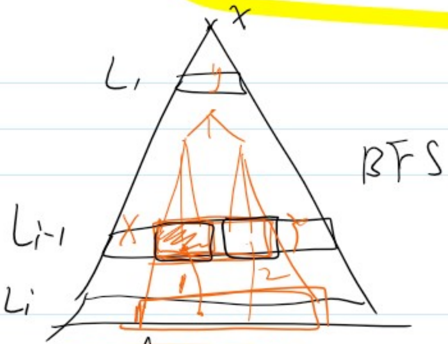
By claim 2, $|L_k| \geq n$, a contradiction. ☒

Lemma 2. (Jiang–Ma, 2018) Let H be a connected graph and let $\varphi: E(H) \to \{1,2\}$ be a function s.t. there is at least one edge colored by i for $\forall i \in \{1,2\}$.
Let H_i be the subgraph of H consisting of all edges colored by i.
If the average degree of H_1 is at least $\boxed{2p+2}$, then then there exists a path of length p in H whose first edge is colored by 2 and all other edges are colored by 1

(pf: EX)

Pf 2 (Jiang–Ma, 2018). We show (not using A-B path lemma)

$$ex(n, C_{2k}) \leq 8kn^{1+1/k} + 6(k-1)n.$$



Similarly define BFS-tree T on the subgraph H of G, where H is bipartite with $\delta(H) \geq 4kn^{1/k} + 3(k-1)$.

Claim 1 $e_H(L_{i-1}, L_i) \leq 4k(|L_{i-1}| + |L_i|)$, $\forall i = 1, 2, \ldots, k$.

Pf. Suppose $e_H(L_{i-1}, L_i) > 4k(|L_{i-1}| + |L_i|)$

$\Rightarrow$ Take a connected component H^* with $d(H^*) \geq 8k$

Take a minimal tree T' with $V(H^*) \cap L_{i-1} \subseteq V(T')$.

Let X be a subset of $V(H^*) \cap L_{i-1}$ containing all vertices of one branch of T'.

$\ldots Y \ldots V(H^*) \ldots L_{\ldots} \setminus X$

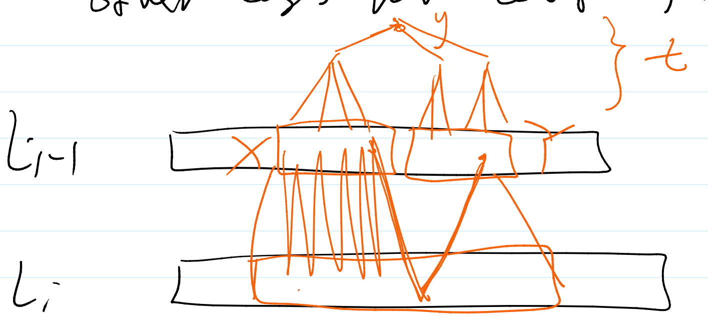
Let X be a subset of $V(H^*) \cap L_{i-1}$ … one branch of T'.

Let $Y = V(H^*) \cap L_{i-1} \setminus X$.

Color all edges in H^* by color 1 if it has an end in X,
or by color 2 if it has an end in Y.

$\Rightarrow d(H_1) + d(H_2) = d(H^*) \geq 8k$

$\Rightarrow$ assume $d(H_1) \geq 4k$. By Lemma 2, $\exists$ a path P of length $\geq 2k$ whose first edge has color 2 and all other edges have color 1.



$\Rightarrow$ we can find consecutive even cycles of lengths $2t+2, 2t+4, 2t+6, \cdots, 2t+2k$ for some $t < i \leq k$

$\Rightarrow \exists$ a cycle of length $2k$, a contradiction $\boxed{G}$

By claim 2 $\Rightarrow$ completing proof. $\boxed{\&}$

The current best bound on $ex(n, C_{2k})$ is as follows.

Thm (Bukh–Jiang, 2016)

$$ex(n, C_{2k}) \leq 80\sqrt{k}\log k \cdot n^{1+1/k} + 10k^2 n$$

Their proof heavily replies on A-B path Lemma.

Conjecture (Erdős–Simonovits) $\forall k \geq 2$

$$ex(n, C_{2k}) = \Theta(n^{1+1/k}).$$

$$ex(n, C_{2k}) = \Theta(n^{1+1/k}).$$

known for $k=2,3,5$ only